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Noise-induced escape from attractors

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Abstract. In this paper, we deal with several aspects of Kautz' minimum principle for the noise-induced escape from attractors: we present a slightly simpler derivation, point out a connection with Lyapunov exponents and invariant manifolds, and generalise to chaotic maps.

1. Introduction

The study of noise-induced escape from attractors has a very old history (for a review, see e.g. [1]). The classic work by Arrhenius was concerned with chemical reactions and dealt with the escape from a potential well induced by thermal noise. His main result was that the escape rate α could be written as a prefactor times a Boltzmann factor:

$$\alpha = B\, \mathrm{e}^{-E\beta} \tag{1}$$

where E is the minimal energy needed to leave the potential well, and $\beta = 1/kT$. While there is a large and ever-growing literature on the correct prefactor B [1], the exponential factor (giving the dominant temperature dependence) is both easy to understand and uncontroversial.

Essentially the same mathematics applies also in those cases far from thermal equilibrium which formally can be described by the motion in some potential function under the influence of Gaussian noise:

$$\dot{x} = -\nabla U(x) + \sigma \xi(t) \tag{2}$$

with σ being a constant and with $\xi(t)$ being white Gaussian noise normalised to

$$\langle \xi_i(t) \xi_k(t') \rangle = 2\delta_{ik}\delta(t-t'). \tag{3}$$

We just have to replace the temperature in (1) by the noise amplitude according to the dissipation-fluctuation theorem, $kT = \sigma^2$, to arrive at

$$\alpha = B\, \mathrm{e}^{-\Delta U/\sigma^2} \tag{4}$$

where ΔU is the potential height of the lowest point of escape (usually a saddle) above the minimum.

For the more general case where the dynamics cannot be described by a static potential or where the noise is not Gaussian, the situation is much less simple. In the

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following, we shall only consider the case where the noise is Gaussian and white, but we shall allow an arbitrary autonomous equation of motion

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) + \sum_{\alpha} \boldsymbol{\sigma}_\alpha \xi_\alpha(t). \tag{5}$$

Here, $\boldsymbol{x}$ and $\boldsymbol{\sigma}_\alpha$ are vectors in some $\mathbf{R}^n$, $\boldsymbol{F}$ is a function $\mathbf{R}^n \to \mathbf{R}^n$, and the ξ_α are normalised as in (3). Notice that this ansatz is sufficiently general to include, e.g., the equation for a Josephson element studied in [2],

$$\beta\ddot{\phi} + \dot{\phi} + \sin\phi = i_0 + \sqrt{\Gamma}\xi(t). \tag{6}$$

This can be written as

$$\dot{\phi} = \chi/\beta \qquad \dot{\chi} = -\chi/\beta - \sin\phi + i_0 + \sqrt{\Gamma}\xi(t) \tag{7}$$

which is obviously of the form of (5). Equation (6) will serve as an application later.

In addition to the flow (5), we will also study the discrete time counterpart

$$\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n) + \sum_{\alpha} \boldsymbol{\sigma}_\alpha \xi_{\alpha n} \tag{8}$$

where $\langle \xi_{\alpha n}\xi_{\beta m}\rangle = 2\delta_{\alpha\beta}\delta_{nm}$.

In this paper, we shall not attempt to compute the prefactor B. Instead, we shall follow closely [2, 3] in computing the exponential factors analogous to the one in (4).

The important result of [2, 3]† was that for the escape from any attractor there exists a minimum principle which allows us to write down an equation of motion for the fastest escape trajectory, and which allows a relatively easy computation of the exponential factor.

It is the main point of the present paper to simplify and clarify somewhat the arguments of [2, 3], and to extend them to the general case embodied in (5, 8) (in [2, 3], only special cases had been studied; we must admit that the extension is rather trivial, and was anticipated in [3]). In particular, we shall show by an example that the method also works if the escape occurs through a fractal basin boundary.

2. The optimal escape path

For an autonomous system, the escape rate is defined via the conditional probability that $\boldsymbol{x}(\Delta t)$ is not in the basin of attraction $\mathscr{A}$, while $\boldsymbol{x}(t')$ was in $\mathscr{A}$ for all $t'<0$,

$$\alpha = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\,\mathrm{Prob}(\boldsymbol{x}(\Delta t)\notin\mathscr{A}\,|\,\boldsymbol{x}(t')\in\mathscr{A},\ t'<0). \tag{9}$$

Using the characteristic function $\chi_{\mathscr{A}}(\boldsymbol{x})$ of the attractor basin, we can write this as a path integral

$$\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\log\left\{\int \mathrm{D}\xi\,\exp\left(-\frac{1}{4}\int_{-\infty}^{t}\mathrm{d}\tau\sum_\alpha \xi_\alpha(\tau)^2\right)\prod_{\tau<t}\chi_{\mathscr{A}}(\boldsymbol{x}(\tau))\right\}\Bigg|_{t=0}. \tag{10}$$

The minimum principle conjectured in [2, 3] results from assuming that the path integral is dominated by the most probable single path. Neglecting all factors except the exponential, we obtain

$$\alpha \simeq \inf_{\xi}\exp\left(-\frac{1}{4}\int_{-\infty}^{0}\mathrm{d}\tau\sum_\alpha \xi_\alpha(\tau)^2\right)\Bigg|_{\boldsymbol{x}(0)\in\partial\mathscr{A}} \tag{11}$$

† See note added in proof.

where $\partial\mathscr{A}$ is the boundary of the basin of attraction. If the infimum is actually attained, the most probable escape path for the noise (called *optimal* in the following) is thus given by that path which minimises the inegral

$$\int_{-\infty}^{0} \mathrm{d}\tau \sum_{\alpha} \xi_\alpha(\tau)^2$$

under the constraint that the trajectory $\boldsymbol{x}(t)$, given by (5), passes through the basin boundary at time $t=0$. If the optimal trajectory reaches the boundary $\partial\mathscr{A}$ only asymptotically, the infimum is not attained in any finite time, and we have to replace the upper time limit by ∞. The discrete time case (8) is completely analogous, with the integral simply replaced by a sum.

The extremal problem is most easily solved by taking (5) into account by means of Lagrangian multipliers. We thus have to minimise the 'Lagrangian'

$$L=\frac{1}{2}\int \mathrm{d}t \sum_{\alpha} \xi_\alpha(t)^2+\int \mathrm{d}t\, \boldsymbol{\eta}(t)\cdot\left[\dot{\boldsymbol{x}}-\boldsymbol{F}(\boldsymbol{x})-\sum_{\alpha} \boldsymbol{\sigma}_\alpha \xi_\alpha(t)\right]. \tag{12}$$

Varying L with respect to $\xi_\alpha(t)$ gives

$$\xi_\alpha(t)=\boldsymbol{\eta}(t)\cdot\boldsymbol{\sigma}_\alpha \tag{13}$$

while varying it with respect to $\boldsymbol{x}(t)$ gives

$$\dot{\eta}_k=\sum_i \frac{\partial F_i}{\partial x_k}\eta_i. \tag{14}$$

Finally, varying it with respect to $\eta_k(t)$ gives back (5). After eliminating ξ_α in favour of $\boldsymbol{\eta}$, this becomes

$$\dot{x}=\boldsymbol{F}(\boldsymbol{x})+\mathbf{C}\boldsymbol{\eta}(t) \tag{15}$$

with the matrix $\mathbf{C}$ given by

$$\mathbf{C}_{jk}=\sum_{\alpha} \sigma_{\alpha j}\sigma_{\alpha k}. \tag{16}$$

Notice that this is just the diffusion matrix in the Fokker–Planck equation corresponding to (5).

Equations (14) and (15) are our main result. They have to be supplemented by the boundary conditions

$$\lim_{t\to-\infty} \boldsymbol{\eta}(t)=0 \qquad \boldsymbol{x}(-\infty)\in \text{attractor} \qquad \boldsymbol{x}(t_f)\in\partial\mathscr{A} \tag{17}$$

where t_f is 0 or ∞ according to whether the boundary $\partial\mathscr{A}$ is reached within finite time or not.

For the discrete time map (8), we obtain analogously

$$\eta_{k,n-1}=\sum_i \frac{\partial f_i(\boldsymbol{x}_n)}{\partial x_{k,n}}\eta_{i,n} \tag{18}$$

and

$$\boldsymbol{x}_{n+1}=\boldsymbol{f}(\boldsymbol{x}_n)+\mathbf{C}\boldsymbol{\eta}_n. \tag{19}$$

By eliminating $\boldsymbol{\eta}$, one could recast (14) and (15) into a single higher-order equation for $\boldsymbol{x}$. For the examples studied in [2, 3], this would lead to the fourth-order equations given there. The advantage of writing the equations of motion in the form of (14) and (15) (respectively (18) and (19)), as compared with the fourth-order form of [2, 3], is mainly the following. They show that the extremal principle gives primarily an equation for the optimal *noise*, and they let us rather easily discuss the behaviour of this noise when $t \to -\infty$ and when $t \to t_f$.

In particular we see that the asymptotic behaviour of $\eta(t)$ for $t \to -\infty$ is governed by the negative of the transpose of the derivative matrix

$$T_{ki} = \frac{\partial F_k}{\partial x_i} \tag{20}$$

which governs the evolution of an infinitesimal distance vector $\delta \boldsymbol{x}$ in the noise-free system. Thus, the initial growth of $|\eta(t)|$ is governed by the negative (stable) Lyapunov exponents of the noise-free system. If there is only one such negative Lyapunov exponent $\lambda < 0$, then the initial growth is $\eta(t) \sim e^{|\lambda| t}$, and the direction of $\boldsymbol{\eta}(t)$ in any point has to be orthogonal to the unstable manifold. Even if there is more than one negative Lyapunov exponent, this severely restricts the solution manifold and greatly simplifies the integration of (14) and (15).

More precisely, assume we use a shooting method with a randomly chosen (but very small) initial vector $\boldsymbol{\eta}(t_0)$. Then all components in the subspace tangent to the unstable manifold will decrease exponentially, and the only relevant parameters are those describing the components orthogonal to it. Thus the parameter space for the shooting method is in general smaller than expected naively.

If there are several negative Lyapunov exponents, then it is not *a priori* clear which one describes the initial growth of $\boldsymbol{\eta}(t)$. Numerical results for a map with an attractor with two different negative Lyapunov exponents will be given in § 3.3.

For maps, the situation is indeed very similar. There, the evolution is governed by the inverse of the transpose of the tangent matrix

$$t_{ki} = \frac{\partial f_k}{\partial x_i} \tag{21}$$

and the consequences for the initial growth of η_n are exactly the same as for flows.

A special situation prevails if the map is not invertible, as e.g. the logistic map $x_{n+1} = a - x_n^2 + \sigma\xi_n$. In this case, there is no stable manifold if the attractor is chaotic. Equation (18) now becomes $\xi_{n-1} = -2x_n\xi_n$, and it is easily checked that the most likely escape to infinity is via a single jump, $\xi_n = (2-a)/\sigma\delta_{n,n_0}$, occurring at $x_n = 0$. The escape rate thus predicted is $\alpha \propto e^{-(2-a)^2/\sigma^2}$. This was already given in [4], where also the prefactor was derived for arbitrary 1D maps, and where it was verified by numerical simulations. The escape from a periodic attractor of the logistic map, on the other hand, involves many steps, with ξ_n increasing initially as $e^{-n\lambda}$ [5].

The behaviour of $\boldsymbol{\eta}(t)$ for $t \to t_f$ depends on whether t_f is 0 or ∞. In the former case, $\boldsymbol{\eta}(t = t_f = 0)$ has to be finite. In the latter case $\boldsymbol{\eta}(t)$ has to decrease exponentially to zero according to the *positive* Lyapunov exponents on the repellor on $\partial\mathcal{A}$ forming the exit 'door' for the optimal escape trajectory. In the next section we shall present examples illustrating both possibilities. We have not been able to predict *a priori* in general which of the two cases holds.

3. Applications

3.1. Potential case

Let us first discuss the potential case with isotropic noise, (equation (2)). One easily verifies that a solution of (14) and (15) is in this case given by

$$\sigma\boldsymbol{\xi}(t) = 2\dot{\boldsymbol{x}}(t) = 2\nabla U(\boldsymbol{x}) \tag{22}$$

so that $\boldsymbol{\xi}^2 = 4\dot{\boldsymbol{x}}\cdot\nabla U(\boldsymbol{x})/\sigma^2 = 4\dot{U}/\sigma^2$. Integrating (11) we immediately recover (4). In addition to (4), the present argument yields also the optimal escape trajectory. It is the time-reversed trajectory of the unperturbed system leading from the saddle on $\partial\mathcal{A}$ to the potential minimum [2]. The escape time t_f is infinite.

3.2. Josephson junction

A very similar argument holds also in the more general case of (6). There, (14) can be written as

$$\beta\ddot{\xi} - \dot{\xi} + \cos\phi\,\xi = 0. \tag{23}$$

A formal solution of (6) and (23) is

$$\sqrt{\Gamma}\xi(t) = 2\dot{\phi}(t). \tag{24}$$

For the parameters considered in [2], (6) has one saddle and two attractors. The latter are one stationary state and one limit cycle [2]. In the case of the limit cycle, we see immediately that (24) cannot be the physical solution since $\dot{\phi}$ does not vanish for $t \to -\infty$ while the correct ξ should. But in the case of the stationary state, (24) *is* the correct solution and it leads exactly to the physical solution found in [2].

In addition to this physical solution, [2] found several unphysical solutions for the escape from the stationary state, and claimed that the selection of the physical solution is not entirely trivial. It is, however, easily seen that these unphysical 'solutions' are *not* solutions on the entire interval $-\infty < t < t_f$, but only on subintervals $t_0 < t < t_f$ and $-\infty < t < t_0$ with some finite t_0, with $\xi(t) \equiv 0$ for $t < t_0$, and with discontinuities of $\xi(t)$ or of $\dot{\xi}(t)$ at t_0. Thus they are not solutions of the extremal problem, corresponding not even to local extrema.

On the limit cycle, we have one negative Lyapunov exponent. Thus a shooting method to solve (14) and (15) needs only one relevant initial parameter, in contrast to what was stated in [2]. As initial condition for $(\phi, \dot{\phi})$ we choose a random point on the limit cycle, by first iterating the noiseless equation for a sufficiently long time. For $(\xi, \dot{\xi})$ we take $\xi = 0$ and $\dot{\xi} = -\varepsilon$ with $\varepsilon \simeq 10^{-6}$-$10^{-5}$.

I verified that the optimal escape occurred through the saddle $(\phi_0, 0)$ as found already in [2]. Assume now that the approach towards it were asymptotic, with $t_f = \infty$. A straightforward linear stability analysis around the saddle shows that ξ would have to vanish faster than exponentially, and $(\phi, \dot{\phi})$ would have to approach the saddle tangentially to its stable manifold. In the opposite case of $t_f = 0$ and $\xi(t_f) = \xi_f \neq 0$, we would have $\phi(t) \simeq \phi_0 + (t_f - t)^2\xi_f/(2\beta)$ for $t \to t_f$, and the saddle is approached vertically in the $(\phi, \dot{\phi})$ plane.

To decide between these two possibilities, I solved the problem numerically for the two cases $i_0 = 0.83$, $\beta = (0.13)^{-2}$, and $i_0 = 0.5$, $\beta = 25$ also studied in [2]. In both cases the second alternative was found to be realised, although in both cases ξ_f was found to be very small ($\xi_f/\xi_{\max} \simeq 10^{-2}$). Thus for $i_0 = 0.5$, $\beta = 25$ the escape trajectory agrees within the thickness of the line with figure 11 of [2] although this figure does not show any crossover to a vertical slope when approaching the saddle. The escape rate can be written as

$$\alpha \propto \mathrm{e}^{-\varepsilon/\Gamma}. \tag{25}$$

In the first case ($i_0 = 0.83$, $\beta = (0.13)^{-2}$) I found $\varepsilon = 14.280$, to be compared with $\varepsilon = 14.4$ found in [2] by means of the minimum principle, to $\varepsilon = 14.5 \pm 1.0$ found in [2] by simulations, and to $\varepsilon = 14.7$ found in [5]. In the second case ($i_0 = 0.5$, $\beta = 25$) I found $\varepsilon = 0.8059$, to be compared with $\varepsilon = 0.81$ found in [2].

3.3. *Map with fractal basin boundary*

As our last example, we consider the map

$$\begin{aligned} x_{n+1} &= x_n^2 - y_n^2 + 0.7x_n + \tfrac{1}{2}\sigma\xi_n \\ y_{n+1} &= 2x_n y_n + 0.7x_n + 0.5y_n + \tfrac{1}{2}\sigma\eta_n . \end{aligned} \tag{26}$$

The noiseless map can be considered as a perturbed quadratic conformal map, and it has essentially the same toplogical features: it has stable fixed points at $\boldsymbol{x} = 0$ and $\boldsymbol{x} = \infty$, and it has a connected Julia set which is topologically equivalent to a circle (figure 1). The Lyapunov exponents at $(0, 0)$ are $\lambda_1 = \log 0.7$ and $\lambda_2 = \log 0.5$, with associated invariant directions $\boldsymbol{v}_1 = (2, 7)$ and $\boldsymbol{v}_2 = (0, 1)$. The basin boundary of this attractor is the Julia set. It is fractal and it contains unstable periodic orbits but no stable ones. It has two different positive Lyapunov exponents $\lambda_1 \simeq \lambda_2 \simeq \log 2$.

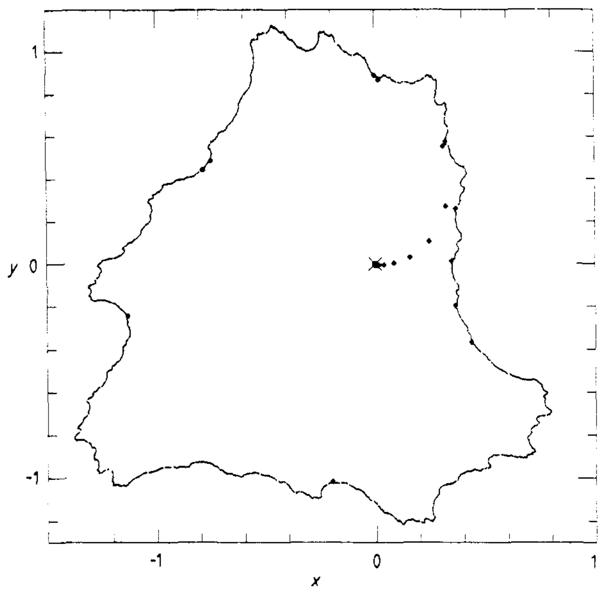

Figure 1. Stable fixed point (cross), repeller (fractal closed curve), and optimal escape path (heavy dots) for (26).

The optimal escape path shown in figure 1 clearly starts off along the x axis, i.e. orthogonal to the vector $\boldsymbol{v}_2$ corresponding to λ_2. Indeed, $|\xi_n|$ grows initially as $n^{-0.7}$, i.e. the growth is governed by λ_1. I have found no *a priori* reason why this should be so. For large times, the optimal escape path seems to approach the Julia set only asymptotically (i.e. $n_f = \infty$), and it does not seem to escape via any of the periodic orbits. Instead, it seems to follow a chaotic orbit on the Julia set. For $n \to \infty$, $|\xi_n|$ decays roughly $\sim 2^{-n}$ as predicted, but accuracy was not sufficient to decide which of the two Lyapunov exponents governed this decay.

The escape rate predicted from the above is $\alpha \propto e^{-\varepsilon/\sigma^2}$ with $\varepsilon = 0.041\,99$. In order to test this, I made simulations with 400 runs for each noise level. The resulting average escape times (which should be the inverses of the escape rates) are shown as diamonds in figure 2. For low noise levels, they agree perfectly with the prediction shown as a straight line (notice that only the slope but not the intercept of the line is predicted).

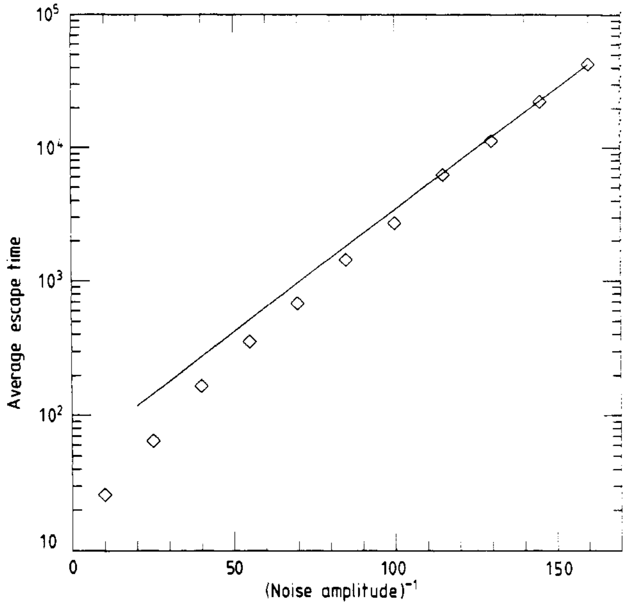

Figure 2. Logarithms of the average escape times against inverse noise levels for the map (26). Diamonds: simulation results from 400 runs for each noise level. Straight line: slope predicted by Kautz' minimum principle.

4. Conclusions

In this paper, we have elaborated on Kautz' [2] (see note added in proof) elegant minimum principle for noise-induced escape from attractors. We have seen that a slight rewriting of the resulting equations of motion allows for an easier discussion of the boundary conditions. This in turn also simplified the numerics, with the result that we could predict more accurately the low-noise limit of the escape rates.

We have seen a close connection to Lyapunov exponents and their associated invariant directions of the noise-free systems. We have also seen that the method works for discrete maps and for systems with fractal basin boundaries, where the optimal escape does not pass through a simple saddle.

I have not attempted in this paper to estimate the prefactor B in the formula for the escape rate. But using the path integral (10) it should not be too difficult to compute it in a saddle point approximation. I hope to come back to this point in a later publication.

Acknowledgments

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Note added in proof. After submitting this paper, I was informed that the minimum principle of [2] had been derived and used before by Graham and Tél [7].

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